Modeling of European option with additional compound interest payments

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Abstract - Solved the problem of pricing the popular European type of securities option in the case of a multivariate binomial financial market consisting of a quantity K bond and a single stock. It is assumed that the bond values are multiplied by the compound interest rate on the payments - more than one number. In the case of the built model, the risk-neutral probability is defined, the generalized formula of fair price coke, Ross and Rubinstein and the buy-sell parity formula are adopted. Recursive formulas for the fair price of an option, two-step binomial trees, and formulas for calculating the minimum hedge are built into the n-step task.

Consider a multifunctional binomial financial market consisting of a quantity of k bonds and 1 stock. Say

$$B_n = B_n^1 + \cdots + B_n^k \qquad (1)$$

Where n= 0,1, ..., N There are moments of time. Assume that, B_n, B_n^1 , ... , B_n^k Bond values multiply by interest at a compound interest rate multiplied accordingl $\lambda_n > 1$, $C_n^i > 1$, $i = 1,2,\dots k$ On the quantities. we have

$$B_n = (1+r)\cdot B_{n-1}\cdot \lambda_n,$$

$$B_n^1 = (1+r_1)B_{n-1}^1\cdot C_n^1 \ , \ \dots,$$

$$B_n^k = (1+r_k)\cdot B_{n-1}^k\cdot C_n^k, \qquad (2)$$

Where r, $r_1, \dots r_k$ Interest rates are complex, so we have the following model of evolution over time in multi-binary financial market bonds.

$$(1+r)\cdot B_{n-1}\cdot\lambda_n = (1+r_1)B_{n-1}^1\cdot C_n^1 + \dots + (1+r_k)\cdot B_{n-1}^k\cdot C_n^k \qquad (3)$$

In this model the interest rate r is unknown, we can easily get that from the third equation

$$r = r_n = \frac{r_1\cdot B_{n-1}^1\cdot C_n^1+\cdots+r_k\cdot B_{n-1}^k\cdot C_n^k + B_{n-1}^1\cdot C_n^1}{(B_{n-1}^1+\cdots+B_{n-1}^k)\cdot\lambda_n} + \cdots + \frac{(B_{n-1}^k\cdot C_n^k - B_{n-1} - \cdots - B_{n-1}^k)\cdot\lambda_n}{(B_{n-1}^1+\cdots+B_{n-1}^k)\cdot\lambda_n} \qquad (4)$$

In case λ and C are equal to one, we will have a multivariate binomial financial market model without considering the coupon.

The second component of the model is action

$$S_n = (1+\rho_n)\, S_{n-1}\,, \quad S_0 > 0, \qquad (5)$$

$S=(S_n)$ is a stock, the interest rate $r > 0$ is constant, And ρ_n is a sequence of random and uniformly distributed random values. However $P(\rho_n=b)=p$, $P(\rho_n=a)=1-p=q$, $-1<a<r<b$. Note that n= 0,1, ..., N are time moments.

Determine also the risk-neutral probability with the following equation

$$p^* = p_n^* = \frac{r_n - a}{b-a} \qquad (6)$$

Imagine an investor who has an X_0 amount at the initial n = 0 moment, his desire to use the financial market to make this amount equal to f_N the amount at the N moment in the future. This desire of the investor is called investment problem. Suppose at moment n the investor has bought a β_n quantity bond and a γ_n quantity stock, i.e. he has an asset portfolio (strategy) $\pi_n = (\beta_n, \gamma_n)$. The corresponding amount for this portfolio is equal to

$$X_n^{\pi} = \beta_n B_n + \gamma_n S_n \quad (7)$$

Where B_n and S_n are at the n moment of time the prices of 1 bond and 1 stock respectively. A portfolio is called a minimum hedge if the equation is completed

$$X_N^{\pi} = \beta_N B_N + \gamma_N S_N = f_N \quad (8)$$

ევროპული ტიპის ყიდვის სტანდარტული ოფციონი f_N გადახდის ფუნქციით ჩაიწერება შემდეგნაირად $f_N = (S_N - K)^+$

Where K is ე. წ. Under the negotiated price, the option buyer can withdraw the option only at the last N moment of time. The task of the option seller-issuer is to build a minimum hedge with the amount received from the sale of the option. Note that the issuer sells the option at a fair price so that it can build a minimum hedge and f_N save money. Thus, the main tasks of option pricing are: to determine the fair price of the option, to construct a minimum hedge, and to determine the appropriate equity process for that hedge.

Finally, we note that the problem of option pricing has been explored in the case of the dual binomial financial market. Which consists of 1 bond and 1 stock.

2. Consider the model (1), (5) And a European-type purchase standard payment option

$$f_N = f(S_N) = (S_N - K)^+ \quad (9)$$

Function, where $(x)^+ = \max(x;0)$ but, K>0 Is the price of the agreement (contractual), ie the issuer is obliged to sell the share at the price K at the moment N of time.

The general formula for the fair price of coke, Ross, and Rubinstein is as follows:

$$C_N = (1+r)^{-N}\mathcal{F}_N(S_0 \; ; \; p^*) = S_0 \sum_{k=k_0}^{N} (p^*)^k (1-p^*)^{N-k} \left(\frac{1+a}{1+r}\right)^N \left(\frac{1+b}{1+a}\right)^k$$

$$-K(1+r)^{-N} \sum_{k=k_0}^{N} C_N^k (p^*)^k (1-p^*)^{N-k} \quad (10)$$

With the formula where $k_0 = k_0(a,b,S_0,K)$ The smallest integer for which it ends

$$S_0 \, (1+a)^N \left(\frac{a+b}{1+a}\right)^{k_0} > K \quad (11)$$

Inequality.

3. Now consider the European type of sales standard payment option

$$f_N = f(S_N) = (K - S_N)^+ \quad (12)$$

By function. In this case the fair P_N price of the option can be found using formula (10). P_N formula is called the buy-sell parity formula

The fair P_N price of the standard European selling option is calculated by the payment function (12)

$$P_N = C_N - S_0 + K \, (1+r)^{-N}. \quad (13)$$

By the formula.

We really have

$$max(0, K - S_N) = max(S_N - K, 0) S_N + K \quad (14)$$

Therefore

$$P_N = E^* \, (1+r)^{-N} \, max(0, K - S_N) = C_N - E^*(1+r)^{-N} S_N + K(1+r)^{-N}. \quad (15)$$

Now, if we notice that $E^* S_N = (1+r)^{-N} S_0$ Then we get the approving (13) formula.

4. To build a minimum hedge, the so-called The principle of the respondent portfolio is as follows: Suppose at moment n of time the corresponding capital $X_n^{\pi} = \beta_n B_n + \gamma_n S_n$ of the investor's portfolio $\pi_n = (\beta_n, \gamma_n)$

We need to build such $\pi_{n+1} = (\beta_{n+1}, \gamma_{n+1})$ portfolio that at moment n of time is its corresponding capital

Must be equal to the moment in time (n + 1)

$$X_{n+1}^{\pi} = \beta_{n+1}B_{n+1} + \gamma_{n+1}S_{n+1} = f\,(S_{n+1}) \quad (16)$$

Of magnitude where $f = f\,(S_N)$ Any payment function, n=0,1,...,N. (1),(5) Unknown in case of model β_{n+1} and γ_{n+1} For the parameters (16) we get from the equation two-dimensional linear equations

$$\beta_{n+1}(1+r)B_{n+1} + \gamma_{n+1}(1+b)S_{n+1} = f((1+b)S_n),$$

$$\beta_{n+1}(1+r)B_n + \gamma_{n+1}(1+a)S_n = f\,((1+a)S_n)$$

A system whose solution β_{n+1}^* and γ_{n+1}^* are given

$$\beta_{n+1}^* = \frac{(1+b)f\,((1+a)S_n) - (1-a)\,f\,((1+b)S_n)}{(1+r)(b-a)B_n}, \quad (17)$$

$$\gamma_{n+1}^* = \frac{f\,((1+b)S_n) - f\,((1+a)S_n)}{(b-a)S_n}, \quad (18)$$

According to the formulas, at moment N we have $\pi_N^* = (\beta_N^*, \gamma_N^*)$ corresponding to the portfolio

$$X_n^{\pi^*} = \beta_N^* B_N + \gamma_N^* S_N = f\,(S_N)$$

Capital.

If we enter the values (17) and (18) into the equation (16) instead of the values β_{n+1} and γ_{n+1}, then we can easily get $X_N^{\pi^*}$ of the capital

$$X_N^{\pi^*} = (1+r)^{-1}[\,p^* f\,((1+b)S_n) + (1-p^*)\,f\,((1+a)S_n)]$$

A representation where the probabilistic representation p^* is defined by the equation (5)

5. The following recurring equations are used to calculate stock prices, payment function values, and fair price of the option:

- At the last, final n = N moment we have the value N+1:
  $f_{N,j} = f\,(S_{N,j})$, j=0,1,...,N

- Finally at the previous n=N-1 moment
  $C_{N-1,j} = (1+r)^{-1}\big[\,p^* f_{N.j+1} + (1-p^*)\,f_{N,j}\big]$, j=0,1,..., N-1.
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- n=N-k at the moment
  $C_{N-k,j} = (1+r)^{-1}\big[\,p^* C_{N-k+1.j+1} + (1-p^*)\,C_{N-k+1,j}\big]$, j=0,1,..., N-k.
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- n=N-N=0 At the moment the option fair is calculated
  $$C_N = C_{0,0} = C(f_N) = (1+r)^{-1}\big[\,p^* C_{1,1} + (1-p^*)\,C_{1,0}\big] \quad (19)$$
  Price.

6. (19) Using equations, they often build so-called equations for visual purposes. Binomial trees. For example, in a two-step option pricing task, a binomial tree would look like this:

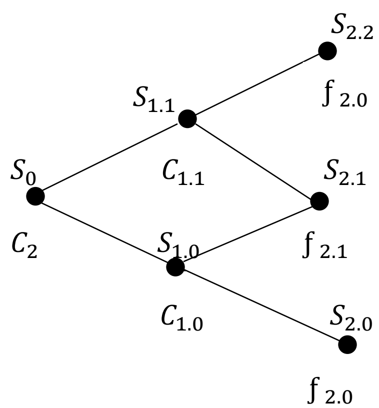


Figure 1.1. Two-step binomial tree

Figure 1.1 shows the following values

$S_{2.0} = S_0(1+a)^2$, $\quad f_{2.0} = f(S_{2.0})$

$S_{2.1} = S_0(1+b)(1+a)$, $\quad f_{2.1} = f(S_{2.1})$

$S_{2.2} = S_0(1+b)^2$, $\quad f_{2.2} = f(S_{2.2})$

$S_{1.0} = S_0(1+a)$,

$S_{1.1} = S_0(1+b)$,

$$C_{1.0} = (1+r)^{-1}\left[p^* f_{2,1} + (1-p^*) f_{2,0}\right] \quad (20)$$

$$C_{1.1} = (1+r)^{-1}\left[p^* f_{2,2} + (1-p^*) f_{2,1}\right] \quad (21)$$

And the fair price of the option

$$C_2 = C(f_2) = (1+r)^{-1}\left[p^* C_{1,1} + (1-p^*) C_{1,0}\right] \quad (22)$$

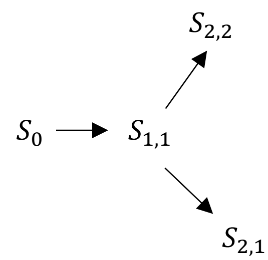
7. European type standard purchase option.

Binomial (B, S) - The initial market data are:

$$B_0 = 20, \quad r = \frac{1}{5}, \quad S_0 = 100,$$

$$a = -\frac{2}{5}, \quad b = \frac{3}{5}, \quad K = 100.$$

Solve the two-step task of buying a standard option and Consider the case



- At time n = 2 we will have the binomial tree in the three final (last) nodes for the possible share prices and payment function values:

$S_{2.0} = S_0(1+b)^0(1+a)^2 = 100(\frac{8}{5}^0)(\frac{3}{5}^2) = 36$,

$S_{2.1} = S_0(1+b)(1+a) = 100\frac{8}{5}\cdot\frac{3}{5} = 96$,

$S_{2.2} = S_0(1+b)^2(1+a)^0 = 100(\frac{8}{5}^2)(\frac{3}{5}^0) = 256$,

$f_{2.0} = f(S_{2.0}) = \max(S_{2.0}-K, 0) = \max(36-100, 0) = 0$,

$f_{2.1} = f(S_{2.1}) = \max(S_{2.1}-K, 0) = \max(96-100, 0) = 0$,

$f_{2.2} = f(S_{2.2}) = \max(S_{2.2}-K, 0) = \max(256-100, 0) = 156$,

At time n = 1 we will have two corresponding nodes:

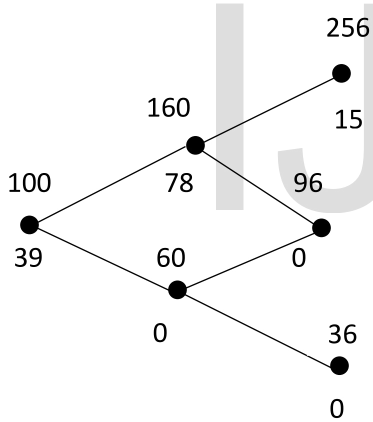
$C_{1.0} = \quad (1+r)^{-1}\left[\, p^{*}f_{2,1} \; + (1-\; p^{*})\, f_{2,2}\right] = \frac{5}{6}\left(\frac{3}{5}\cdot 0 + \frac{2}{5}\cdot 0\right) = 0$

$C_{1.1} = (1+r)^{-1}\left[\, p^{*}f_{2,2} \; + (1-\; p^{*})\, f_{2,1}\right] = \frac{5}{6}\left(\frac{3}{5}\cdot 156 + \frac{2}{5}\cdot 0\right) = 78$

And at a fair price

$C_2 = (1+r)^{-1}\left[\, p^{*}C_{1,1} \; + (1-\; p^{*})\, C_{1,0}\right] = \frac{5}{6}\left(\frac{3}{5}\cdot 78 + \frac{2}{5}\cdot 0\right) = 39.$

Thus, a two-step binomial tree will have the following appearance:



- we have:

$$\beta_1^{*} = \frac{(1+b)C_{1,0} - (1+a)\, C_{1,1}}{(1+r)(b+a)B_0} = \frac{\frac{8}{5}\cdot 0 - \frac{3}{5}\cdot 78}{\frac{6}{5}\cdot 1\cdot 20} = -\frac{39}{20},$$

$$\gamma_1^{*} = \frac{C_{1,1} - C_{1,0}}{(b-a)S_0} = \frac{78-0}{1\cdot 1000} = \frac{39}{50},$$

So we got it $\pi_1^{*} = (\beta_1^{*}, \gamma_1^{*}) = \left(-\frac{39}{20}, \frac{39}{50}\right)$ The corresponding capital of this portfolio at the moment n = 0.

$$X_0^{\pi^{*}} = \beta_1^{*}B_0 + \gamma_1^{*}S_0 = -\frac{39}{20}\cdot 20 + \frac{39}{50}\cdot 100 = 39$$

Then we have

$$\beta_2^{*} = \frac{(1+b)C_{1,2} - (1+a)\, f_{2,2}}{(1+r)(b-a)B_1} = \frac{\frac{8}{5}\cdot 0 - \frac{3}{5}\cdot 156}{\frac{6}{5}\cdot 1\cdot 24} = -\frac{13}{4},$$

$$\gamma_2^{*} = \frac{f_{2.2} - f_{1.1}}{(b-a)S_{1.1}} = \frac{156-0}{1\cdot 160} = \frac{39}{40},$$

So we got it $\pi_1^{*} = (\beta_1^{*}, \gamma_1^{*}) = \left(-\frac{13}{4}, \frac{39}{40}\right)$ .The corresponding capital of this portfolio at the moment n =1.

$$X_1^{\pi^{*}} = \beta_2^{*}B_1 + \gamma_2^{*}S_{1.1} = -\frac{13}{4}\cdot 24 + \frac{39}{40}\cdot 160 = 78$$

Then we have

$$\beta_2^{*} = \frac{(1+b)C_{1,2} - (1+a)\, f_{2,2}}{(1+r)(b-a)B_1} = \frac{\frac{8}{5}\cdot 0 - \frac{3}{5}\cdot 156}{\frac{6}{5}\cdot 1\cdot 24} = -\frac{13}{4},$$

$$\gamma_2^{*} = \frac{f_{2.2} - f_{1.1}}{(b-a)S_{1.1}} = \frac{156-0}{1\cdot 160} = \frac{39}{40},$$

IX.I. Suppose, $S_2 = S_{2,2} = 256$ The price of the bond

$$B_2 = (1+r)\cdot B_1 = \frac{6}{25}\cdot 24 = \frac{144}{5}$$

In this case

$$X_2^{\pi^{*}} = -\frac{13}{4}\cdot\frac{144}{5} + \frac{39}{40}\cdot 256 = 156$$

Which exactly meets the obligation of the option $f_{2,2} = f(S_{2,2}) = 156$, The issuer will sell $\frac{39}{40}$ shares and receive an amount equal to $\frac{39}{40}\cdot 256 = \frac{1248}{5}$. From this amount he will repay the bond debt of $\frac{13}{4}$ or an amount equal to $\frac{13}{4}\cdot\frac{144}{5} =$

$\frac{468}{5}$, and equal to the remaining $\frac{1248}{5}$ - $\frac{468}{5}$ = 156

Will fulfill the option obligation with the amount:

$\mathfrak{f}_{2,2} = 156$

IX.II. Suppose $S_2 = S_{2,1}$=96, In this case

$X_2^{\pi^*} = -\frac{13}{4} \cdot \frac{144}{5} + \frac{39}{40} \cdot 96 = 0$.

Which exactly meets the obligation of the option

$\mathfrak{f}_{2,1} = \mathfrak{f}(S_{2,1}) = 0$,

The issue will sell $\frac{39}{40}$ shares and will receive an amount equal to $\frac{39}{40} \cdot 96 = \frac{468}{5}$. From this amount he will pay $\frac{13}{4}$ of the bond debt or an amount equal to $\frac{13}{4} \cdot \frac{144}{5} = \frac{468}{5}$, and he will pay nothing on the option liability because $\mathfrak{f}_{2,1} = 0$.

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